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ESTIMATING COMMON PARAMETERS OF DIFFERENT CONTINUOUS DISTRIBUTIONS

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A B S T R A C T

Estimating a common parameter is the most essential and quite fascinating task across various probability distributions. This article addresses the challenge of estimating this parameter through the application of Maximum Likelihood Estimation (MLE). Numeric determination of common parameters is conducted for several distributions, including the Lomax distribution, Gamma distribution, Rayleigh distribution, and Weibull distribution. In cases where distributions lack a closed-form solution, estimation of MLEs is achieved using the Newton-Raphson technique. Furthermore, asymptotic confidence intervals are computed utilizing the Fisher information matrix tailored to each distribution. The performance evaluation of these estimators centers on the assessment of bias and mean squared error. To enable a numerical comparison of these estimators, the Monte Carlo simulation method is employed. Finally, these techniques are applied to real-time rainfall data to assess parameter estimates for each distribution.

1. INTRODUCTION

Measuring the entire population would be too difficult, parameters serve as descriptive measures of the population as a whole. Consequently, we resort to estimating parameters by selecting a sample from the population since we lack knowledge of their exact values. Various popular methods for parameter estimation include Bayes estimation, least square estimation, method of moments, uniformly minimal variance unbiased estimation (UMVUE) method, and others. However, owing to its distinctive characteristics, the most effective and renowned method for parameter estimation is the Maximum Likelihood (ML) approach.

The British statistician, geneticist, and eugenicist R.A. Fisher, often referred to as the father of statistics, demonstrated (Aldrich, 1977) that the method of moments may not be effective when calculating the parameters of Pearson Type III distributions. He recommended applying the MLE approach instead. MLE is a straightforward technique for obtaining an estimate of an unknown parameter. A new distribution named as Generalized Exponential (GE) distribution introduced by (Gupta, 2001) estimated the unknown parameters of GE(α, λ) using MLE and compared with other estimation methods. While the estimation of parameters for various continuous distributions has been tackled individually by different authors in distinct ways, the comparative estimation of parameters for different distributions

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has received little attention. The primary objective of this study is to estimate the common parameter of different populations in order to determine which distribution provides the best estimates for the collected rainfall data, considering their bias and mean squared error.

In this study, common parameters are estimated for a variety of distributions, including the Lomax, Rayleigh, Weibull, and Gamma distributions. For each distribution, maximum likelihood estimators are determined numerically, and for those that lack a closed form, the MLEs are estimated using the Newton-Raphson technique. Asymptotic confidence intervals have also been calculated using the Fisher information matrix of each distribution. Let's consider these distributions one by one.

Lomax distribution:

The Lomax distribution, first proposed by K.S. Lomax in 1954 for lifetime data analysis and modelling business failure data, is a special type of Pareto distribution, also known as the Pareto Type II distribution. Widely applied in various contexts, the Lomax distribution is known for its heavy-tailed characteristics. According to Hassan and Al Ghamdi, the Lomax distribution proves useful in reliability modelling and life testing problems. This distribution has unquestionably served as a model for various datasets in numerous studies. Harris, for instance, used the Lomax distribution for income and wealth data in 1978. When dealing with heavily tailed data, the Lomax distribution is preferred over the exponential distribution. Moreover, it finds numerous applications across a range of disciplines, including biology, business, economics, actuarial modelling, queuing theory, and reliability modelling.

Let us consider two independent Lomax populations with a common scale parameter 'δ' and different shape parameters σ_1 and σ_2 respectively. Let $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be m and n random samples taken from two Lomax Populations $L(\delta, \sigma_1)$ and $L(\delta, \sigma_2)$ respectively. Here $L(\delta, \sigma_1)$ denotes the Lomax population having the probability density function.

$$f(x;\delta,\sigma_1) = \frac{\sigma_1}{\delta}\left[1 + \frac{x}{\delta}\right]^{-(\sigma_1+1)}; x \geq 0,\ \sigma_1, \delta > 0 \quad (1.1)$$

likewise, $L(\delta, \sigma_2)$ denotes the Lomax population which has the probability density function,

$$f(y;\delta,\sigma_2) = \frac{\sigma_2}{\delta}\left[1 + \frac{y}{\delta}\right]^{-(\sigma_2+1)}; y \geq 0,\ \sigma_2, \delta > 0 \quad (1.2)$$

Giles (2013) discussed MLE for the parameters of the Lomax distribution and alternative techniques for reducing this bias when the sample size is small. Hasanain (2022) delved into the parameter estimation of the Lomax distribution using three different loss functions, employing both MLE and Bayesian estimation methods. The Lindley approximation was utilized to obtain the best estimates. To draw conclusions about the parameters of a Lomax distribution, (He, 2023) established objective Bayesian techniques. Al- Zahrani and Sobhi (2013) estimated the parameters of the Lomax distribution under general progressive censoring by considering the probability density function of the two-parameter Lomax distribution.

Rayleigh distribution:

Among all probability distributions, the Rayleigh distribution is one of the most frequently used. Introduced by Lord Rayleigh in 1880, it generally appears as a special case of the Weibull distribution. The Rayleigh distribution finds significant applications in statistical communication theory and target theory. It is widely used in reliability analysis, applied statistics, and clinical investigations, all of which are extremely important fields. This distribution is in fact, a specific example of the Weibull distribution with two parameters scale and location.

Let us consider two independent Rayleigh populations with a common scale parameter λ and different location parameters μ_1 and μ_2 respectively. $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be m and n random samples taken from two Rayleigh Populations $Ray(\lambda, \mu_1)$ and $Ray(\lambda, \mu_2)$ respectively. Here $Ray(\lambda, \mu_1)$ denotes the Rayleigh population having the probability density function,

$$f(x;\lambda,\mu_1) = 2\lambda(x - \mu_1)e^{-\lambda(x-\mu_1)^2};\ x > 0, \mu_1, \lambda > 0 \quad (1.3)$$

likewise, $Ray(\lambda, \mu_2)$ denotes the population which has the probability density function,

$$f(y;\lambda,\mu_2) = 2\lambda(y - \mu_2)e^{-\lambda(y-\mu_2)^2};\ y > 0, \mu_2, \lambda > 0 \quad (1.4)$$

Dey et al. (2014) estimated the parameters of the Rayleigh distribution using various methods, including MLE, method of moments, L-moment estimators, least squares estimators, weighted least squares estimators, percentile-based estimators, and Bayes estimators, all for a single sample (Dey and Dey, 2012) estimated Bayesian estimators and calculated Prediction Intervals for a Rayleigh Distribution using conjugate prior. N. Balakrishnan, in 1989, derived the Approximate MLE of the Scale Parameter of the Rayleigh Distribution with Censoring (Kundu, 2005). They also derived the parameters of the Generalized Rayleigh distribution using different estimation methods. Johnson, Kotz, and Balakrishnan in 1994 briefly discussed the Rayleigh distribution with two parameters. In addition, (Bhat, 2023) formulated a new lifetime probability model named the Power Rayleigh distribution, and the unknown parameters are estimated using MLE.

Weibull distribution:

The Weibull distribution was introduced by Mr. Waloddi Weibull in 1937. It is an adaptable distribution that can take on the characteristics of other types of distributions based on the shape parameter value. This distribution is particularly useful for analyzing life statistics and determining the reliability of items. The Weibull distribution finds extensive applications in life data analysis and reliability analysis due to its versatility, which is of paramount importance. It is primarily employed to model the range of behaviors for a given function, depending on the parameter values. The distribution function is typically well described by the probability density function. The Weibull distribution method represents one of the distinctive ways to analyze real-world data significantly. Several approaches are typically employed to assess the reliability of the data in a specific manner.

Let us consider two independent Weibull populations with a common scale parameter 'α' and different shape parameters β_1 and β_2 respectively. $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be m and n random samples taken from two Weibull Populations Weibull(α, β_1) and Weibull(α, β_2) respectively. Here Weibull(α, β_1) denotes the Weibull population having the probability density function,

$$f(x;\alpha,\beta_1) = \beta_1 \alpha^{-\beta_1} x^{-\beta_1 - 1} e^{-\left(\frac{x}{\alpha}\right)^{\beta_1}}; x \geq 0, \alpha, \beta_1 > 0 \quad (1.5)$$

likewise, Weibull(α, β2) denotes the population which has the probability density function

$$f(y;\alpha,\beta_2) = \beta_2 \alpha^{-\beta_2} y^{-\beta_2 - 1} e^{-\left(\frac{y}{\alpha}\right)^{\beta_2}}; y \geq 0, \alpha, \beta_2 > 0 \quad (1.6)$$

(Tan, 2009) in the year came to the conclusion that there is no analytical solution for the restricted MLE of the scale parameter for a given shape parameter, and he finally developed a new approach that is thought to be more effective and efficient at handling interval data than regular MLE methods, directly developed EM algorithms, as well as genetic algorithms. Stone (1977) used the maximum likelihood method to find the Weibull distribution's parameters, and they also gave a method for calculating confidence intervals. As can be seen from an example, the confidence intervals for sample sizes often used in dielectric life tests can be wide. Lai (2011) provided comprehensive explanations of a variety of extensions, parameter estimation techniques, and fundamental Weibull distribution characteristics that are useful for modelling complex data sets. N. Balakrishnan and Kateri proposed an alternative approach based on a simple and easy-to-apply graphical method (Balakrishnan and Kateri (2008) also readily shows the existence and uniqueness of the maximum likelihood estimators. Yang et al (Yang, 2007) proposed a new approach called Modified MLE (MMLE) In the case of complete and Type II censored data, the bias of the MLE can be substantial. This is noticeable even when the sample size is large.

Gamma distribution:

The Gamma distribution is a widely used and versatile statistical distribution in fields such as Business, Science, Reliability, Modelling, and Climate Analysis Centre (CAC), among others. It was first discovered by James Clerk Maxwell and later developed by Ludwig Boltzmann. In 2021, Eric U., Oti Michael, Olusola, and Francis studied the properties and applications of the Gamma distribution in real-life situations.

Let us take two independent irregular samples from two Gamma populations with a common scale parameter 'η' and different shape parameters ρ_1 and ρ_2 respectively. $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be m and n random samples taken from two gamma populations Gamma(η, ρ_1) and Gamma(η, ρ_2) respectively. Here Gamma(η, ρ_1) denote the gamma population having the probability density function,

$$f(x;\eta,\rho_1) = \frac{1}{\Gamma\rho_1 \eta^{\rho_1}} x^{\rho_1 - 1} e^{-\frac{x}{\eta}}; x > 0, \eta > 0, \rho_1 > 0 \quad (1.7)$$

likewise, Gamma(η, ρ_2) denotes the population which has the probability density function

$$f(y;\eta,\rho_2) = \frac{1}{\Gamma\rho_2 \eta^{\rho_2}} y^{\rho_2 - 1} e^{-\frac{y}{\eta}}; y > 0, \eta > 0, \rho_2 > 0. \quad (1.8)$$

Several studies have attempted to address the problem of estimating parameters of the Gamma distribution. Chapman (Hirose, 1995), S. C. Choi and R. Wette (Choi, 1969), Cohen and Whitten (Wilks, 1990), Daniel S. Wilks and Hideo Hirose (Nagamani, 2017) in 2017 have applied the Maximum Likelihood Estimation (MLE) method to estimate parameters of the Gamma distribution. In addition to calculating the common shape parameter (Tripathy, 2017), they also estimated the parameters for the Gamma distribution using MLE (Husak, 2007). Husak (Shenton, 1969) estimated the MLEs of the Gamma distribution for monthly rainfall data in Africa. David E. Giles and Hui Feng (Shenton, 1969) demonstrated how the methodology suggested by Cox and Snell in 1968 can be easily used to construct a closed-form adjustment to these MLEs.

2. MAXIMUM LIKELIHOOD ESTIMATION AND THE ASYMPTOTIC CONFIDENCE INTERVALS OF VARIOUS CONTINUOUS DISTRIBUTIONS

In this part, we examine our model and provide a numerical approach for calculating the maximum likelihood estimate of the parameters, as well as asymptotic 95% confidence intervals for the parameters of Lomax, Weibull, Rayleigh, and Gamma distributions individually.

2.1 MLE and Asymptotic confidence intervals of "Lomax distribution":

Let $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be two independent random samples taken from Lomax(δ, σ_1)

In this sub section 2.1 we are going to estimate the maximum likelihood estimates of Lomax distribution as well as the confidence intervals for its parameters.

Let us consider the joint probability density function of X and Y is,

$$f(x, y; \delta, \sigma_1, \sigma_2) = \frac{\sigma_1^m \sigma_2^n}{\delta^{m+n}} \times \prod_{i=1}^{m}\left[1 + \frac{x_i}{\delta}\right]^{-(\sigma_1+1)} \times \prod_{j=1}^{n}\left[1 + \frac{y_j}{\delta}\right]^{-(\sigma_2+1)}$$

here $\sigma_1, \sigma_2 > 0$, $\delta > 0$ and x, y > 0.

The log-likelihood function of f (x, y) is given by,

$$L(x, y; \delta, \sigma_1, \sigma_2) = m log\sigma_1 + n log\sigma_2 - (\sigma_1 + 1) \sum_{i=1}^{m} log\left[1 + \frac{x_i}{\delta}\right] - (\sigma_2 + 1)\sum_{j=1}^{n} log\left[1 + \frac{y_j}{\delta}\right] - (m + n)\, log\, \delta$$

the maximum value of L(x, y; δ, σ_1, σ_2) can be obtained by differentiating with respect to δ , σ_1 and σ_2 and equating to zero. Then solving for δ, σ_1 and σ_2 we get the MLEs. After certain calculations the system of three nonlinear equations are obtained as follows.

$$-(m + n) + \sum_{i=1}^{m}\left[\frac{x_i(\sigma_1+1)}{\delta+x_i}\right] + \sum_{j=1}^{n}\left[\frac{y_j(\sigma_2+1)}{\delta+y_j}\right] = 0$$

$$\frac{m}{\sigma_1} - \sum_{i=1}^{m} log\left[1 + \frac{x_i}{\delta}\right] = 0$$

$$\frac{n}{\sigma_2} - \sum_{j=1}^{n} log\left[1 + \frac{y_j}{\delta}\right] = 0$$

next, we derive the information matrix and hence the expression for asymptotic variance of the MLEs of the parameters associated to our model.

We have the log-likelihood function for our model as,

$$L(x, y; \delta, \sigma_1, \sigma_2) = m log\sigma_1 + n log\sigma_2 - (\sigma_1 + 1) \sum_{i=1}^{m} log\left[1 + \frac{x_i}{\delta}\right] - (\sigma_2 + 1)\sum_{j=1}^{n} log\left[1 + \frac{y_j}{\delta}\right] - (m + n)\, log\, \delta$$

we denote $\theta_1 = \sigma_1$, $\theta_2 = \sigma_2$ and $\theta_3 = \delta$. The Fisher information matrix for our model is after certain calculations, we obtain the Fisher information matrix as follows.

$$I(\sigma_1, \sigma_2, \delta) = (I_{ij}), \quad i, j = 1,2,3$$

where $I_{ij} = E(P_{ij})$,$P_{ij} = -\frac{\partial^2 L}{\partial\theta_i\, \partial\theta_j}$

$$I(\sigma_1, \sigma_2, \delta) = \begin{bmatrix} \frac{-m}{\sigma_1^2} & 0 & \frac{m-\delta\sum_{i=1}^{m} t_1}{\delta} \\ 0 & \frac{-n}{\sigma_2^2} & \frac{n-\delta\sum_{j=1}^{n} t_1}{\delta} \\ \frac{m-\delta\sum_{i=1}^{m} t_1}{\delta} & \frac{n-\delta\sum_{j=1}^{n} t_1}{\delta} & \frac{L_1}{\delta^2} \end{bmatrix}$$

here, $L_1 = m + n - (\sigma_1 + 1)(m - \delta^2 \sum t_1^2) - (\sigma_2 + 1)(n - \delta^2 \sum t_1^2)$, $t_1 = \frac{1}{\delta+x_i}$ and $t_2 = \frac{1}{\delta+y_j}$

then, the inverse of this matrix is obtained as,

$$I(\sigma_1, \sigma_2, \delta)^{-1} = \begin{bmatrix} a_{11} & \frac{\sigma_1^2\sigma_2^2(m-a\delta)(n-b\delta)}{D_1} & \frac{\sigma_1^2 n\delta(m-a\delta)}{D_1} \\ \frac{\sigma_1^2\sigma_2^2(m-a\delta)(n-b\delta)}{D_1} & a_{22} & \frac{\sigma_2^2 m\delta(n-b\delta)}{D_1} \\ \frac{\sigma_1^2 n\delta(m-a\delta)}{D_1} & \frac{\sigma_2^2 m\delta(n-b\delta)}{D_1} & \frac{\delta^2 mn}{D_1} \end{bmatrix}$$

here, $a_{11} = \frac{-\sigma_1^2[d\sigma_2^2\delta^2 - 2n\sigma_2^2 b\delta + \sigma_2^2 n^2 + Bn]}{D_1}$,

$a_{22} = \frac{-\sigma_2^2[c\sigma_1^2\delta^2 - 2ma\delta\sigma_1^2 + \sigma_1^2 m^2 + Bm]}{D_1}$,

$D_1 = (\sigma_1^2 a^2\delta^2 n - 2mn\sigma_1^2 a\delta + \sigma_1^2 m^2 n + \sigma_2^2 b^2\delta^2 m - 2mn\sigma_2^2 b\delta + \sigma_2^2 mn^2 + Bmn)$,

$a = \sum t_1$, $b = \sum t_2$, $c \sum {t_1}^2$, $d = \sum {t_2}^2$ and

$$B = m + n - (\sigma_1 + 1)\left(m - \delta^2 \sum t_1^2\right) - (\sigma_2 + 1)\left(n - \delta^2 \sum t_2^2\right)$$

by using the above information, we can easily calculate the asymptotic 95% confidence intervals for the parameters δ, σ_1 and σ_2. The 95% confidence interval for θ_j is obtained as,

$$\theta_j \pm 1.96\sqrt{\left(I(\theta)_{jj}\right)^{-1}}$$

the 95% confidence interval for the common scale parameter δ, shape parameters σ_1 and σ_2 are estimated as follows,

$$\hat{\delta}_{ML} \pm 1.96\sqrt{\frac{\hat{\delta}_{ML}^2 mn}{D_1}},$$

$$\hat{\sigma}_{1ML} \pm 1.96\sqrt{a_{11}} \quad \text{and} \quad \hat{\sigma}_{2ML} \pm 1.96\sqrt{a_{22}}$$

2.2 MLE and Asymptotic confidence intervals of "Rayleigh distribution":

Let $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be independent random samples taken from Ray(λ, μ_1) and Ray(λ, μ_2) (as given in 1.3 and 1.4) respectively, with common scale parameter λ and different Shape parameters μ_1 and μ_2. In this sub section 2.2 we are going to estimate the maximum likelihood estimates of Rayleigh distribution as well as the confidence intervals for its parameters.

The joint probability density function of X and Y is obtained as,

$$f(x,y;\lambda,\mu_{\{1\}},\mu_{\{2\}}) = (2\lambda)^{\{m+n\}} \prod_{\{i=1\}}^{\{m\}}(x_i-\mu_1) \prod_{\{j=1\}}^{\{n\}}(y_j-\mu_2)\, e^{-\lambda\left(\sum_{i=1}^{m}(x_i-\mu_1)^2+\sum_{j=1}^{n}(y_j-\mu_2)^2\right)}$$

t
he log-likelihood function of f(x, y) is given by,

$$L(x,y;\lambda,\mu_1,\mu_2) = (m+n)log(2\lambda) + \sum_{i=1}^{m} log(x_i-\mu_1) \sum_{j=1}^{n} log(y_j-\mu_2) - \lambda\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{j=1}^{n}(y_j-\mu_2)^2\right)$$

the maximum value of $L(x,y;\lambda,\mu_1,\mu_2)$ can be obtained by differentiating with respect to μ_1, μ_2 and λ and equating to zero. Then solving for λ, μ_1 and μ_2 we get the MLEs. After certain calculations the system of three nonlinear equations are obtained as follows,

$$\hat{\lambda} = \frac{m+n}{\sum_{i=1}^{m}(x_i-\mu_1)^2+\sum_{j=1}^{n}(y_j-\mu_2)^2}$$

$$\frac{-1}{\sum_{i=1}^{m}(x_i-\mu_1)} + 2\lambda\sum_{i=1}^{m}(x_i-\mu_1) = 0$$

$$\frac{-1}{\sum_{j=1}^{n}(y_j-\mu_2)} + 2\lambda\sum_{j=1}^{n}(y_j-\mu_2) = 0$$

next, we derive the information matrix and hence the expression for asymptotic variance of the MLEs of the parameters associated to our model. We have the log-likelihood function for our model as,

$$L(x,y;\lambda,\mu_1,\mu_2) = (m+n)log(2\lambda) + \sum_{i=1}^{m} log(x_i-\mu_1) \sum_{j=1}^{n} log(y_j-\mu_2) - \lambda\left(\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{j=1}^{n}(y_j-\mu_2)^2\right)$$

we denote $\theta_1 = \mu_1$, $\theta_2 = \mu_2$ and $\theta_3 = \lambda$.

The Fisher information matrix for our model is obtained as,

$$I(\mu_1,\mu_2,\lambda) = (I_{ij}), \qquad i,j = 1,2,3, \qquad I_{ij} = E(P_{ij})$$

$$P_{ij} = -\frac{\partial^2 L}{\partial\theta_i\,\partial\theta_j}$$

after certain calculations, we obtain the Fisher information matrix as follows,

$$I(\mu_1,\mu_2,\lambda) = \begin{bmatrix} (-2ma\lambda-1)/a & 0 & -2b \\ 0 & (-2nc\lambda-1)/c & -2d \\ -2b & -2d & \frac{-(m+n)}{\lambda^2} \end{bmatrix}$$

here, $a = \sum_{i=1}^{m}(x_i-\mu_1)^2$, $b = \sum_{i=1}^{m}(x_i-\mu_1)$,

$$c = \sum_{j=1}^{n}(y_j-\mu_2)^2 \text{ and } d = \sum_{j=1}^{n}(y_j-\mu_2)$$

the inverse of the matrix $I(\mu_1,\mu_2,\lambda)$ is obtained as,

$$I(\mu_1,\mu_2,\lambda)^{-1} = \begin{bmatrix} b_{11} & -\frac{4\lambda^2abcd}{D_1} & -\frac{2ab\lambda^2[2cn\lambda+1]}{D_1} \\ -\frac{4\lambda^2abcd}{D_1} & b_{22} & -\frac{2cd\lambda^2[2am\lambda+1]}{D_1} \\ -\frac{2ab\lambda^2[2cn\lambda+1]}{D_1} & -\frac{2cd\lambda^2[2am\lambda+1]}{D_1} & b_{33} \end{bmatrix}$$

here, $b_{11} = -\frac{a[2cn^2\lambda-4c\lambda^2d^2+2cmn\lambda+m+n]}{D_1}$,

$b_{22} = -\frac{c[2am^2\lambda-4a\lambda^2b^2+2amn\lambda+m+n]}{D_1}$,

$b_{33} = -\frac{\lambda^2[2am\lambda+1][2cn\lambda+1]}{D_1}$,

$$D_1 = 4\lambda^2ab^2 + 4\lambda^2cd^2 - 2m^2a\lambda - 2n^2c\lambda - 2amn\lambda - 2cmn\lambda - 4\lambda^2acmn^2 - 4\lambda^2acm^2n + 8\lambda^3ab^2cn + 8\lambda^3acd^2m - m - n.$$

Using above information, we can easily calculate the asymptotic 95% confidence intervals for the parameters λ, μ_1 and μ_2. The 95% confidence interval for θ_j is obtained as,

$$\theta_j \pm 1.96\sqrt{\left(I(\theta)_{jj}\right)^{-1}}$$

the 95% confidence intervals for the common rate parameter λ and shape parameters μ_1 and μ_2 are estimated respectively as follows.

$\hat{\lambda}_{ML} \pm 1.96\sqrt{b_{33}}$,

$\hat{\mu}_{1ML} \pm 1.96\sqrt{b_{11}}$ and $\hat{\mu}_{2ML} \pm 1.96\sqrt{b_{22}}$.

2.3 MLE and Asymptotic Confidence Intervals of "Weibull Distribution":

Let $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$ be independent random samples taken from Weibull(α, β_1) and Weibull(α, β_2)(as given in1.5 and 1.6) respectively, with common scale parameter α and different shape parameters β_1 and β_2. In this sub section 2.3 we are going to estimate the maximum likelihood estimates of Weibull distribution as well as the confidence intervals for its parameters.

Let us consider the joint probability density function of X and Y is,

$$f(x,y;\,\alpha,\beta_1,\beta_2) = {\beta_1}^m{\beta_2}^n\alpha^{-(m\beta_1+n\beta_2)} \times \left(\prod_{i=1}^{m}{x_i}^{\beta_1-1}\right)\left(\prod_{j=1}^{n}{y_j}^{\beta_2-1}\right)e^{-\left\{\frac{\sum_{i=1}^{m}x_i^{\beta_1}}{\alpha^{\beta_1}}-\frac{\sum_{j=1}^{n}y_j^{\beta_2}}{\alpha^{\beta_2}}\right\}}$$

the log-likelihood function of f(x, y) is given by,

$$L(x,y;\alpha,\beta_1,\beta_2) = mlog\beta_1 + nlog\beta_2 - (m\beta_1+n\beta_2)log\alpha + (\beta_1-1)\sum_{i=1}^{m}logx_i + (\beta_2-1)\sum_{j=1}^{n}logy_j - \left\{\frac{\sum_{i=1}^{m}x_i^{\beta_1}}{\alpha^{\beta_1}} + \frac{\sum_{j=1}^{n}y_j^{\beta_2}}{\alpha^{\beta_2}}\right\}$$

the maximum value of $L(x, y; \alpha, \beta_1, \beta_2)$ can be obtained by differentiating with respect to α , β_1 and β_2 and equating to zero. Then solving for α , β_1 and β_2 we get the MLEs. After certain calculations the system of three nonlinear equations are obtained as follows.

$$\beta_1 \sum_{i=1}^{m} x_i^{\beta_1} \alpha^{-\beta_1} + \beta_2 \sum_{j=1}^{n} y_i^{\beta_2} \alpha^{-\beta_2} - m\beta_1 - n\beta_2 = 0$$

$$\frac{m}{\beta_1} - mlog\alpha - \sum_{i=1}^{m} logx_i - \left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)^{\beta_1} log\left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)$$

$$\frac{n}{\beta_2} - nlog\alpha - \sum_{j=1}^{n} logy_j - \left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)^{\beta_2} log\left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)$$

next, we derive the information matrix and hence the expression for asymptotic variance of the MLEs of the parameters associated to our model. We have the log-likelihood function for our model as,
we have the log-likelihood function for our model as,

$$L(x, y; \alpha, \beta_1, \beta_2) = mlog\beta_1 + nlog\beta_2$$
$$-(m\beta_1 + n\beta_2)log\alpha + (\beta_1 - 1)\sum_{i=1}^{m} logx_i$$
$$+(\beta_2 - 1)\sum_{j=1}^{n} logy_j - \left\{\frac{\sum_{i=1}^{m} x_i^{\beta_1}}{\alpha^{\beta_1}} + \frac{\sum_{j=1}^{n} y_j^{\beta_2}}{\alpha^{\beta_2}}\right\}$$

we denote θ1 $=\beta_1$, θ2 $= \beta_2$ and θ3 $= \alpha$. The Fisher information matrix for our model is obtained as,

$$I(\beta_1, \beta_2, \alpha) = (I_{ij}), \sim i, j = 1,2,3, \quad I_{ij} = E(P_{ij})$$

$$P_{ij} = -\frac{\partial^2 L}{\partial\theta_i \, \partial\theta_j}$$

after certain calculations, we obtain the Fisher information matrix as follows,

$$I(\alpha, \beta_1, \beta_2) = \begin{bmatrix} \frac{k_1}{\beta_1^2} & 0 & \frac{k_2}{\alpha} \\ 0 & \frac{k_4}{\beta_2^2} & \frac{k_3}{\alpha} \\ \frac{k_2}{\alpha} & \frac{k_3}{\alpha} & \frac{k_5}{\alpha^2} \end{bmatrix}$$

Here,

$$k_1 = -m - \beta_1^2\left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)^{\beta_1}\left(\log\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)^2,$$

$$k_2 = -m + \left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)^{\beta_1}\left[\beta_1 \log\left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right) + 1\right],$$

$$k_3 = -n + \left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)^{\beta_2}\left[\beta_2 \log\left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right) + 1\right],$$

$$k_4 = -n - \beta_2^2\left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)^{\beta_2}\left(\log\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)^2,$$

$$k_5 = m\beta_1 + n\beta_2 - \beta_1(\beta_1 + 1)\left(\frac{\sum_{i=1}^{m} x_i}{\alpha}\right)^{\beta_1} - \beta_2(\beta_2 + 1)\left(\frac{\sum_{j=1}^{n} y_j}{\alpha}\right)^{\beta_2}$$

the inverse of this matrix $I(\alpha, \beta_1, \beta_2)$ is obtained as,

$$I(\alpha, \beta_1, \beta_2)^{-1} = \begin{bmatrix} \frac{-\beta_1^2(k_4k_5 - k_3^2\beta_2^2)}{D_1}, & \frac{-\beta_1^2 k_2 k_3 \beta_2^2}{D_1} & \frac{-\beta_1^2 k_2 k_4 \alpha}{D_1} \\ \frac{-\beta_1^2 k_2 k_3 \beta_2^2}{D_1} & \frac{-\beta_2^2(k_1k_3 - k_2^2\beta_1^2)}{D_1} & \frac{k_1 k_3 \beta_2^2 \alpha}{D_1} \\ \frac{-\beta_1^2 k_2 k_4 \alpha}{D_1} & \frac{k_1 k_3 \beta_2^2 \alpha}{D_1} & \frac{-k_1 k_4 \alpha^2}{D_1} \end{bmatrix} \quad (2.38)$$

Here, $D_1 = k_1k_4k_5 - \beta_2^2 k_1 k_3^2 - \beta_1^2 k_2^2 k_4$.

By using above information, we can easily calculate the asymptotic 95% confidence intervals for the parameters $\alpha, \beta_1 and\ \beta_2$. The 95% confidence interval for θ_j is obtained as,

$$\theta_j \pm 1.96\sqrt{\left(I(\theta)_{jj}\right)^{-1}}$$

the 95% confidence intervals for the parameters α, β_1 and β_2 are as follows

$$\hat{\alpha}_{ML} \pm 1.96\sqrt{\frac{-k_1k_4\hat{\alpha}_{ML}^2}{D_1}}$$

$$\hat{\beta}_{1ML} \pm 1.96\sqrt{\frac{-\hat{\beta}_{1ML}^2\left(k_4k_5 - k_3^2\hat{\beta}_{2ML}^2\right)}{D_1}}$$

$$\hat{\beta}_{2ML} \pm 1.96\sqrt{\frac{-\hat{\beta}_{2ML}^2\left(k_1k_3 - k_2^2\hat{\beta}_{1ML}^2\right)}{D_1}}$$

2.4 MLE and Asymptotic Confidence Intervals of "Gamma Distribution":

Let $X = (X_1, X_2 \cdots, X_m)$ and $Y = (Y_1, Y_2 \cdots, Y_n)$be independent random samples taken from Gamma(ρ_1, η) and Gamma(ρ_2, η) (as given in 1.7 and 1.8) respectively, with common scale parameter η and different Shape parameters ρ_1 and ρ_2. In this sub section 2.4 we are going to estimate the maximum likelihood estimates of Gamma distribution as well as the confidence intervals for its parameters.

The joint probability density function of X and Y is obtained as,

$$f(x, y|\rho_1, \rho_2, \eta) = \frac{(\Pi_{i=1}^{m} x_i)^{\rho_1 - 1}(\Pi_{j=1}^{n} y_j)^{\rho_2 - 1}}{(\Gamma\rho_1)^m(\Gamma\rho_2)^n \eta^{m\rho_1 + n\rho_2}} \times e^{-\frac{1}{\eta}\left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j\right)}$$

the log-likelihood function of f(x, y) is given by,

$$L(x, y; \rho_1, \rho_2, \eta) = (\rho_1 - 1)\sum_{i=1}^{m} log\, x_i + n\, log(\Gamma\rho_2)$$
$$(\rho_2 - 1)\sum_{j=1}^{n} log\, y_j - m\, log(\Gamma\rho_1) -$$
$$-(m\rho_1 + n\rho_2)\, log\, \eta - \frac{1}{\eta}\left(\left\{\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j\right)\right\}$$

the maximum value of $L(x, y; \rho_1, \rho_2, \eta)$ can be obtained by differentiating with respect to ρ_1, ρ_2, and η and equating to zero. Then solving for ρ_1, ρ_2, and η we get the MLEs. After certain calculations the system of three nonlinear equations are obtained as follows,

$$\sum_{\{i=1\}}^{\{m\}} \log x_i - m\psi(\rho_1) - m\, log\, \eta = 0$$

$$\sum_{\{j=1\}}^{\{n\}} \log y_j - n\psi(\rho_2) - n\, log\, \eta = 0$$

$$\eta(m\,\rho_1 + n\rho_2) - \sum_{\{i=1\}}^{\{m\}} x_i + \sum_{\{j=1\}}^{\{n\}} y_j = 0$$

Where,

$\psi(\rho_1) = \frac{d}{d\rho_1}(log\, \Gamma\, \rho_1) \quad and \quad \psi(\rho_2) = \frac{d}{d\rho_2}(log\, \Gamma\, \rho_2)$

are known as the digamma functions.

next, we derive the information matrix and hence the expression for asymptotic variance of the MLEs of the parameters associated to our model. We have the log-likelihood function for our model as,

$$L(x, y; \rho_1, \rho_2, \eta) = (\rho_1 - 1)\sum_{i=1}^{m} log\, x_i + n\, log(\Gamma\rho_2) \\ (\rho_2 - 1)\sum_{j=1}^{n} log\, y_j - m\, log(\Gamma\rho_1) - \\ -(m\rho_1 + n\rho_2)\, log\, \eta - \frac{1}{\eta}\left(\left\{\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j\right)\right\}$$

we denote $\theta 1 = \rho_1$, $\theta 2 = \rho_2$ and $\theta 3 = \eta$. The Fisher information matrix for our model is obtained as,

$$I(\rho_1, \rho_2, \eta) = (I_{ij}), \qquad i, j = 1,2,3, \qquad I_{ij} = E(P_{ij})$$

$$P_{ij} = -\frac{\partial^2 L}{\partial\theta_i\, \partial\theta_j}$$

after certain calculations, we obtain the Fisher information matrix as follows.

$$I(\rho_1, \rho_2, \eta) = \begin{bmatrix} m\Psi'(\rho_1) & 0 & \frac{m}{\eta} \\ 0 & n\Psi'(\rho_2) & \frac{n}{\eta} \\ \frac{m}{\eta} & \frac{n}{\eta} & \frac{m\rho_1 + n\rho_2}{\eta^2} \end{bmatrix}$$

where ψ' (ρ1) and ψ′ (ρ2) are the derivative of digamma function known as tri gamma functions. The inverse of this matrix is obtained as,

$$I(\rho_1, \rho_2, \eta)^{-1} = \begin{bmatrix} \frac{\Psi'(\rho_2)(m\rho_1 + n\rho_2) - n}{mD} & \frac{1}{D} & -\frac{\eta\Psi'(\rho_2)}{D} \\ \frac{1}{D} & \frac{\Psi'(\rho_1)(m\rho_1 + n\rho_2) - m}{nD} & \frac{\eta\Psi'(\rho_1)}{D} \\ -\frac{\eta\Psi'(\rho_2)}{D} & \frac{\eta\Psi'(\rho_1)}{D} & \frac{\eta^2\Psi'(\rho_1)\Psi'(\rho_2)}{D} \end{bmatrix}$$

by using above information, we can easily calculate the asymptotic 95% confidence intervals for the parameters η, ρ_1 and ρ_2.

The 95% confidence interval for θ_j is obtained as,

$$\theta_j \pm 1.96\sqrt{\left(I(\theta)_{jj}\right)^{-1}}$$

the 95% confidence intervals for the common rate parameter η and shape parameters ρ1, ρ2 are obtained respectively as follows.

$$\hat{\eta}_{ML} \pm 1.96\sqrt{\frac{\hat{\eta}_{ML}^2\Psi'(\hat{\rho}_{1ML})\Psi'(\hat{\rho}_{2ML})}{D}}$$

$$\hat{\rho}_{1ML} \pm 1.96\sqrt{\frac{(m\hat{\rho}_{1ML} + n\hat{\rho}_{2ML})\Psi'(\hat{\rho}_{2ML}) - n}{mD}}$$

$$\hat{\rho}_{2ML} \pm 1.96\sqrt{\frac{(m\hat{\rho}_{1ML} + n\hat{\rho}_{2ML})\Psi'(\hat{\rho}_{1ML}) - m}{nD}}$$

using the Monte-Carlo simulation approach in R programming, all of the above-mentioned estimators are numerically compared in section-3 in terms of bias and mean squared error.

3. NUMERICAL COMPARISONS

In this research, we address the problem of estimating parameters for two similar continuous probability distributions, specifically the Lomax, Weibull, Rayleigh, and Gamma distributions, using the Maximum Likelihood Estimation (MLE) method. As the closed form of the ML estimates does not exist, we are going to use a numerical technique named as Newton-Raphson method to find approximate ML estimates of the common parameter by solving the system of equations of different distributions. Furthermore, we compute the 95% confidence intervals with the help of Fisher information matrix. To compare these estimators numerically, we evaluate their performance in terms of bias and mean squared error. For this purpose, we generate 10,000 random samples for each distribution and compute the bias and mean squared error of each parameter.

By taking the different values of parameters we computed the Estimates, from the table 3.1 to 3.4 represents equal sample sizes of both the samples of different continuous probability distributions from 10 to 50 the ML-estimates of all parameters with its Bias and mean squared error are reported. From table 3.5 to 3.8 having different combination of sample sizes from 10 to 50 with unequal sample sizes, the ML-estimates of all parameters with its bias and mean squared error are computed. From Table 3.9 to 3.12 are the values of 95% Asymptotic confidence intervals for the distributions of Lomax, Weibull, Rayleigh and Gamma distributions respectively, with various different values of common parameter are computed.

1. MLE provided consistent and efficient parameter values for every distribution considered in the study.
2. From the simulation data, it can be said that increasing the sample size reduced both the bias and mean square error for each estimator.
3. Larger sample sizes generally resulted in more accurate parameter estimates with lower standard error.
4. With small sample sizes, the MLE estimates had higher variability and larger standard errors.
5. It was observed that, for fixed sample sizes and fixed shape parameters, the common scale parameter of these distributions increased and its mean squared error values decreased.
6. All the parameter values lay inside the confidence intervals, and the length of confidence intervals for all parameters decreased as the sample size grew.

7. The width of the confidence intervals depended on the sample size and the desired level of confidence.
8. As the sample size increased, the width of the confidence intervals decreased, indicating increased precision in parameter estimation.
9. Based on the specific context and characteristics of the data, it can be analyzed that every distribution has its own individuality and identity.
10. Our simulation analysis yielded comparable results for other combinations of sample sizes and parameters.

Table 3.1. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (10,10).

Θ ↓	Lomax(θ=σ₁, σ₂, δ)			Weibull(θ=β₁, β₂, α)			Rayleigh(θ=μ₁, μ₂, λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.26	0.26	0.76	1.11	0.11	0.11	1.19	0.19	0.26	1.21	0.21	0.26
2	1.77	-0.22	0.46	2.25	0.25	0.32	2.40	0.40	0.44	2.45	0.45	0.48
1.5	1.41	-0.09	0.44	1.49	-0.00	0.05	1.35	-0.14	0.26	1.35	-0.14	0.27
1	1.43	0.43	0.39	1.10	0.10	0.11	1.22	0.22	0.24	1.21	0.21	0.24
2	1.93	-0.06	0.09	2.37	0.37	0.71	2.39	0.39	1.18	2.47	0.47	1.24
2.5	2.35	-0.14	1.48	2.50	-0.00	0.15	2.22	-0.22	0.53	2.25	-0.20	0.74
1	1.24	0.24	0.78	1.10	0.10	0.12	1.21	0.21	0.25	1.20	0.20	0.24
2	2.18	0.18	0.15	2.30	0.30	0.53	2.28	0.28	0.61	2.46	0.46	1.21
3.5	3.37	-0.12	0.42	3.46	-0.03	0.29	3.34	-0.15	0.63	3.15	-0.34	1.43

Table 3.2. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (20,20).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.23	0.23	0.60	1.04	0.04	0.03	1.11	0.11	0.25	1.08	0.08	0.08
2	1.79	-0.20	0.45	2.14	0.14	0.20	2.20	0.20	0.33	2.21	0.21	0.34
1.5	1.37	-0.12	0.39	1.49	-0.00	0.02	-0.05	0.11	0.19	1.42	-0.07	0.13
1	1.34	0.34	0.27	1.05	0.05	0.04	1.11	0.11	0.09	1.09	0.09	0.08
2	1.97	-0.02	0.03	2.12	0.12	0.19	2.21	0.21	0.36	2.22	0.22	0.37
2.5	2.38	-0.11	1.33	2.49	-0.00	0.02	-0.18	0.34	0.22	2.35	-0.14	0.34
1	1.18	0.18	0.74	1.04	0.04	0.03	2.31	-0.18	0.34	1.10	0.10	0.09
2	2.16	0.16	0.13	2.16	0.16	0.18	2.24	0.24	0.48	2.22	0.22	0.35
3.5	3.41	-0.08	0.35	3.49	-0.00	0.00	3.28	-0.22	0.42	3.30	-0.19	0.75

Table 3.3. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (30,30).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.18	0.18	0.43	1.03	0.03	0.02	1.09	0.09	0.20	1.06	0.06	0.04
2	1.81	-0.18	0.36	2.08	0.08	0.10	2.13	0.13	0.21	2.13	0.13	0.21
1.5	1.42	-0.07	0.28	1.49	-0.00	0.01	1.45	-0.04	0.08	1.45	-0.04	0.09
1	1.15	0.15	0.24	1.03	0.03	0.02	1.05	0.05	0.08	1.06	0.06	0.04
2	1.98	-0.01	0.01	2.08	0.08	0.11	2.13	0.13	0.20	2.14	0.14	0.21
2.5	2.41	-0.08	1.27	2.49	-0.00	0.04	2.41	-0.08	0.22	0.241	-0.08	0.25
1	1.14	0.14	0.68	1.03	0.03	0.02	1.07	0.07	0.04	1.08	0.08	0.05
2	2.11	0.11	0.10	2.09	0.09	0.11	2.13	0.13	0.19	2.18	0.18	0.24
3.5	3.48	-0.00	0.33	3.49	-0.00	0.09	3.41	-0.08	0.38	3.31	-0.18	0.51

Table 3.4. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (50,50).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.04	0.04	0.08	1.02	0.02	0.01	1.05	0.05	0.07	1.03	0.03	0.02
2	1.91	-0.08	0.22	2.04	0.04	0.05	2.06	0.06	0.11	2.08	0.08	0.10
1.5	1.48	-0.01	0.07	1.49	-0.00	0.01	1.47	-0.02	0.05	1.46	-0.03	0.05
1	1.12	0.12	0.13	1.01	0.01	0.01	1.04	0.04	0.05	1.08	0.08	0.07
2	1.98	-0.01	-0.00	2.06	0.06	0.05	2.08	0.08	0.11	2.18	0.18	0.34
2.5	2.47	-0.02	1.18	2.49	-0.00	0.02	2.43	-0.06	0.14	2.36	-0.11	0.37
1	1.09	0.09	0.45	1.01	0.01	0.01	1.03	0.03	0.02	1.04	0.04	0.02
2	2.04	0.04	0.08	2.05	0.05	0.05	2.07	0.07	0.09	2.10	0.10	0.10
3.5	3.49	-0.00	0.25	3.50	-0.00	0.04	3.47	-0.02	0.02	3.39	-0.10	0.29

Table 3.5. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (10,20).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.21	0.21	0.24	1.05	0.05	0.06	1.31	0.31	0.89	1.61	0.16	0.17
2	1.79	-0.20	0.21	2.19	0.19	0.22	1.98	-0.01	0.02	0.29	0.29	0.44
1.5	1.42	-0.07	0.05	1.50	-0.00	0.02	1.54	0.04	0.16	1.35	-0.14	0.15
1	1.39	0.39	0.29	1.06	0.06	0.08	1.07	0.07	0.04	1.12	0.12	0.14
2	1.82	-0.17	0.14	2.19	0.19	0.18	1.86	-0.13	0.18	2.15	0.15	0.29
2.5	2.41	-0.08	0.11	2.51	0.01	0.06	2.46	-0.13	0.19	2.39	-0.10	0.36
1	1.22	0.22	0.19	1.10	0.10	0.12	1.07	0.07	0.12	1.06	0.06	0.10
2	1.94	-0.05	0.14	2.12	0.12	0.15	2.04	0.04	0.09	2.32	0.32	0.57
3.5	3.34	-0.15	0.27	3.54	-0.04	0.18	3.47	-0.02	0.04	3.30	-0.19	0.94

Table 3.6. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (20,10).

Θ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.18	0.18	0.45	1.05	0.05	0.04	1.14	0.14	0.74	1.12	0.12	0.11
2	1.82	-0.17	3.1	2.28	0.28	0.54	2.07	0.07	0.26	2.29	0.29	0.66
1.5	1.34	-0.15	0.22	1.49	-0.00	0.04	1.57	0.07	0.25	1.39	-0.1	0.19
1	1.34	0.34	0.28	1.04	0.04	0.03	1.15	0.15	0.42	1.12	0.12	0.10
2	1.94	-0.05	0.19	2.26	0.26	0.53	1.96	-0.03	0.14	2.34	0.34	0.71
2.5	2.39	-0.10	0.14	2.48	0.17	0.13	2.48	-0.01	0.02	2.32	-0.17	0.48
1	1.14	0.14	0.22	1.05	0.05	0.04	1.12	0.12	0.49	1.10	0.10	0.10
2	2.11	0.11	0.17	2.3	0.30	0.57	1.98	-0.01	0.01	2.30	0.30	0.63
3.5	3.42	-0.07	0.15	3.47	-0.02	0.23	3.45	-0.04	0.24	3.32	-0.17	0.99

Table 3.7. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (30,50).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.14	0.14	0.32	1.03	0.03	0.02	1.04	0.04	0.38	1.04	0.04	0.03
2	1.95	-0.04	3.98	2.05	0.05	0.05	1.95	-0.05	0.15	2.09	0.09	0.12
1.5	1.43	-0.06	0.24	1.49	-0.00	0.11	1.47	-0.02	0.06	1.46	-0.03	0.06
1	1.13	0.13	0.29	1.02	0.02	0.02	1.21	0.21	0.78	1.05	0.05	0.03
2	1.97	-0.02	0.08	2.04	0.04	0.05	1.88	-0.11	0.96	2.09	0.09	0.12
2.5	2.41	-0.08	0.19	2.49	-0.00	0.00	2.43	-0.06	0.73	2.44	-0.05	0.18
1	1.11	0.11	0.39	1.02	0.02	0.02	1.27	0.27	0.85	1.04	0.04	0.03
2	2.05	0.05	0.07	2.05	0.05	0.06	1.87	-0.12	0.41	2.08	0.08	0.12
3.5	3.45	-0.04	0.08	3.48	-0.00	0.05	3.38	-0.11	0.24	3.41	-0.08	0.35

Table 3.8. Comparing Biases and MSE of Several Estimators of Several distributions For The different parameters when (m,n) = (50,30).

Θ ↓	Lomax(θ=σ₁,σ₂,δ)			Weibull(θ=β₁,β₂,α)			Rayleigh(θ=μ₁,μ₂,λ)			Gamma(ρ₁, ρ₂, η)		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	1.06	0.06	0.08	1.02	0.02	0.01	1.18	0.18	0.43	1.04	0.04	0.03
2	1.88	-0.11	0.12	2.10	0.10	0.10	1.92	-0.07	0.39	2.11	0.11	0.17
1.5	1.44	-0.05	0.08	1.49	-0.00	0.00	1.41	0.08	0.48	1.45	-0.04	0.07
1	1.11	0.11	0.27	1.02	0.02	0.01	1.22	0.22	0.88	1.04	0.04	0.03
2	1.97	-0.02	0.02	2.08	0.08	0.10	1.87	-0.12	0.76	2.11	0.11	0.15
2.5	2.48	-0.01	0.04	2.49	-0.00	0.04	2.31	-0.28	1.03	2.41	-0.08	0.18
1	1.08	0.08	0.14	1.01	0.01	0.01	1.27	0.27	0.97	1.05	0.05	0.03
2	2.10	0.10	0.31	2.09	0.09	0.10	1.92	-0.07	0.97	1.22	0.11	0.16
3.5	3.48	-0.01	0.01	3.49	-0.00	0.08	3.37	-0.12	0.39	3.37	-0.12	0.38

The Asymptotic Confidence intervals for the parameters of lomax, Weibull, Rayleigh and Gamma Distributions with its parameter values and various combinations of sample sizes are computed in the following tables.

Table 3.9. The 95% confidence intervals of Gamma Distribution for the parameters (ρ_1, ρ_2, η).

(m,n)	Conf($\rho_1 = 2$)	Conf($\rho_2 = 3$)	Conf($\eta = 1.5$)	Conf($\rho_1 = 2$)	Conf($\rho_2 = 3$)	Conf($\eta = 2.5$)
(10,10)	(0.55,4.38)	(1.77,5.60)	(-0.57,3.2)	(0.45,4.55)	(1.75,5.85)	(0.19,4.29)
(15,15)	(0.98,3.58)	(2.10,4.76)	(0.11,2.70)	(0.95,3.61)	(2.09,4.74)	(1.01,3.66)
(30,30)	(1.32,2.98)	(2.39,4.05)	(0.60,2.27)	(1.31,2.93)	(2.42,4.03)	(1.58,3.20)
(50,50)	(1.47,2.71)	(2.51,3.75)	(0.84,2.08)	(1.47,2.65)	(2.52,3.70)	(1.85,3.03)
(10,15)	(0.77,3.98)	(1.94,5.15)	(-0.20,2.98)	(0.79,3.92)	(1.96,5.09)	(0.74,3.87)
(15,10)	(0.80,3.89)	(1.97,5.06)	(-0.14,2.94)	(0.91,3.75)	(2.11,4.94)	(0.89,3.73)
(30,50)	(1.36,2.86)	(2.42,3.92)	(0.70,2.20)	(1.38,2.82)	(2.42,3.86)	(1.77,3.15)
(50,30)	(1.39,2.80)	(2.47,3.87)	(0.76,2.16)	(1.41,2.77)	(2.48,3.84)	(1.75,3.11)

Table 3.10. The 95% confidence intervals of Weibull Distribution for the parameters (β_1,β_2,α).

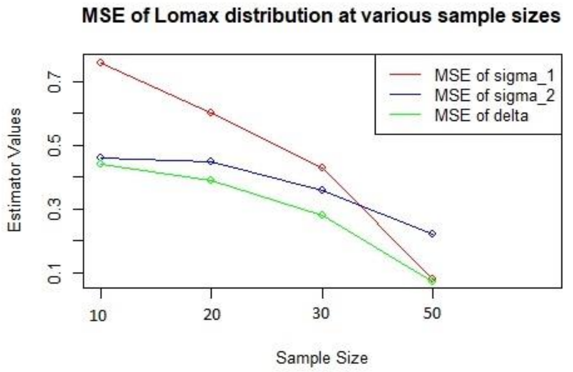
(m,n)	Conf($\beta_1 = 2$)	Conf($\beta_2 = 3$)	Conf($\alpha = 1.5$)	Conf($\beta_1 = 2$)	Conf($\beta_2 = 3$)	Conf($\alpha = 2.5$)
(10,10)	(0.85 , 3.58)	(1.29 , 5.53)	(1.20 , 1.78)	(0.83 , 3.57)	(1.24 , 5.61)	(1.98 , 2.97)
(15,15)	(1.16 , 3.12)	(1.67 , 4.87)	(1.26 , 1.72)	(1.19 , 3.10)	(1.64 , 4.91)	(2.12 , 2.87)
(30,30)	(1.44 , 2.68)	(2.19 , 4.04)	(1.33, 1.65)	(1.45 , 2.70)	(2.18 , 4.00)	(2.23 , 2.76)
(50,50)	(1.59 , 2.52)	(2.37 , 3.76)	(1.37 , 1.61)	(1.59 , 2.56)	(2.38 , 3.75)	(2.29 , 2.69)
(10,15)	(0.91 , 3.50)	(1.74 , 4.77)	(1.25 , 1.73)	(0.89 , 3.61)	(1.66 , 4.80)	(2.07 , 2.90)
(15,10)	(1.10 , 3.21)	(1.26 , 5.53)	(1.23 , 1.76)	(1.08 , 3.23)	(1.21 , 5.77)	(2.03 , 2.93)
(30,50)	(1.46 , 2.67)	(2.35 , 3.82)	(1.36 , 1.62)	(1.45 , 2.68)	(2.36 , 3.76)	(2.28 , 2.71)
(50,30)	(1.58 , 2.50)	(2.17 , 4.05)	(1.354 , 1.64)	(1.60 , 2.50)	(2.16 , 4.10)	(2.24 , 2.75)

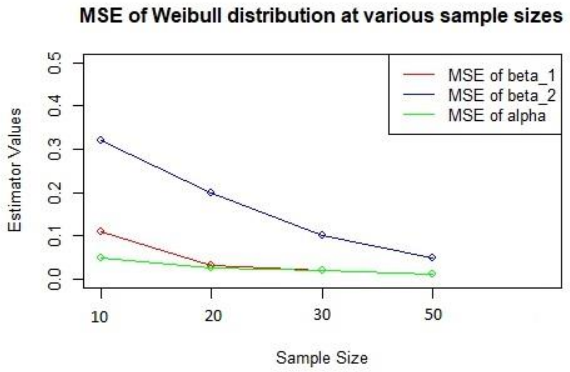
Table 3.11. The 95% confidence intervals of Lomax Distribution for the parameters (σ_1,σ_2,δ).

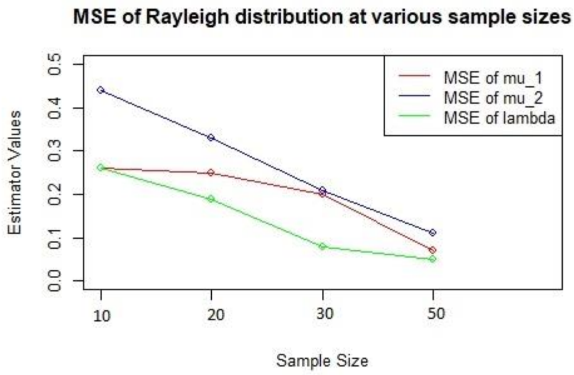
(m,n)	Conf($\sigma_1 = 2$)	Conf($\sigma_2 = 3$)	Conf($\delta = 1.5$)	Conf($\sigma_1 = 2$)	Conf($\sigma_2 = 3$)	Conf($\delta = 2.5$)
(10,10)	(-0.30 , 3.63)	(-5.06 , 5.73)	(-1.48 , 10.61)	(-0.11 , 4.38)	(-4.79 , 5.79)	(0.66 , 7.44)
(15,15)	(0.04 , 3.48)	(-5.33 , 5.77)	(-1.47 , 10.59)	(0.23, 4.17)	(-5.15 , 5.80)	(0.75 , 7.53)
(30,30)	(0.55 , 3.19)	(-5.65 , 5.83)	(-1.46 , 10.58)	(0.78 , 3.94)	(-5.55 , 5.84)	(0.84 , 7.52)
(50,50)	(0.83 , 2.98)	(-5.79 , 5.86)	(-1.47 , 10.61)	(1.21 , 3.75)	(-5.79 , 5.87)	(0.86 , 7.51)
(10,15)	(1.08 , 3.49)	(-5.79 , 5.86)	(-1.21 , 9.83)	(0.62 , 4.13)	(-5.44 , 5.83)	(0.79 , 7.50)
(15,10)	(-1.75 , 3.75)	(-3.77 , 5.48)	(-1.60 , 10.96)	(-0.62 , 4.36)	(-4.43 , 5.73)	(0.54 , 7.77)
(30,50)	(1.39 , 3.72)	(-5.88 , 5.88)	(-1.08 , 9.46)	(1.32 , 3.82)	(-5.84 , 5.87)	(0.89 , 7.42)
(50,30)	(-2.02 , 3.61)	(-3.69 , 5.43)	(-1.72 , 11.36)	(0.16 , 3.84)	(-5.22 , 5.82)	(0.57 , 8.17)

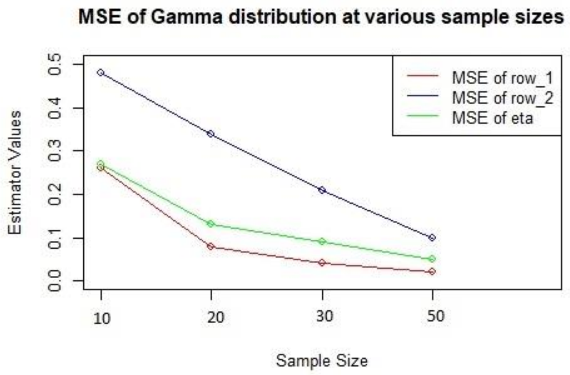
Table 3.12. The 95% confidence intervals of Rayleigh Distribution for the parameters (μ_1,μ_2,λ).

(m,n)	Conf($\mu_1 = 2$)	Conf($\mu_2 = 3$)	Conf($\lambda = 1.5$)	Conf($\mu_1 = 2$)	Conf($\mu_2 = 3$)	Conf($\lambda = 2.5$)
(10,10)	(0.61, 3.36)	(0.25 , 3.90)	(-0.01 , 6.16)	(-0.09 , 6.62)	(0.19 , 3.91)	(1.95 , 4.15)
(15,15)	(0.75 , 2.81)	(0.17 , 3.92)	(0.01 , 6.07)	(0.56 , 5.25)	(0.14, 3.92)	(1.97 , 4.10)
(30,30)	(0.73 , 2.53)	(0.11 , 3.93)	(0.03 , 6.02)	(0.99 , 4.51)	(0.10 , 3.94)	(1.99 , 4.05)
(50,50)	(0.70 , 2.48)	(0.10 , 3.94)	(0.03 , 6.00)	(1.20 , 4.09)	(0.08 , 3.94)	(2.00 , 4.02)
(10,15)	(0.71, 3.01)	(0.22 , 3.91)	(0.00 , 6.11)	(0.36 , 5.70)	(0.17 , 3.92)	(1.96 , 4.11)
(15,10)	(0.72 , 3.04)	(0.20 , 3.91)	(-0.01 , 6.14)	(0.32 , 5.75)	(0.15 , 3.92)	(1.96 , 4.13)
(30,50)	(0.73 , 2.50)	(0.17 , 3.93)	(0.03 , 6.00)	(1.13 , 4.22)	(0.09, 3.94)	(2.00 , 4.03)
(50,30)	(0.74 , 2.50)	(0.10 , 3.94)	(0.03 , 6.01)	(1.12 , 4.28)	(0.08, 3.94)	(1.99 , 4.04)









3.1 Example

In addition, by taking real rainfall data of the Union territory "Andaman and Nicobar Islands" in the span of 60 years, the rain fall data of one population of 30 years from 1911-1940 and another population of 30 years from 1971-2000 are taken as data of two populations, here in the collected data the rain fall is measured in millimeters of rain fell over the year, there is huge difference occurs from month to month as well as year to year in the rain fall data.

For this data the parameters are estimated as well as the bias and mean squared error values are calculated for each distribution.

Table 3.13. Comparing Biases and MSE of Several Estimators of Several distributions for real data.

Θ ↓	Lomax($\theta=\sigma_1,\sigma_2,\delta$)			Weibull($\theta=\beta_1,\beta_2,\alpha$)			Rayleigh($\theta=\mu_1,\mu_2,\lambda$)			Gamma((ρ_1,ρ_2,η))		
	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse	Mle	Bias	Mse
1	4.21	3.21	10.32	6.29	5.29	27.99	0.98	-0.02	0.04	16.02	15.02	225.64
2	4.12	2.12	4.50	6.14	4.14	17.20	1.84	-0.15	0.23	15.28	13.28	176.42
1.5	7.70	-1.5	2.25	245	243.97	59524.1	1.41	-0.08	0.15	14.62	13.21	172.23

By Observing the above values of all the parameters, its bias and mean square error, For the considered real Rain fall data, the Rayleigh distribution is the best fit when compared to the remaining distributions like Lomax, Weibull and Gamma distributions.

The Rayleigh distribution holds potential for various future applications, including:

a) Signal and Image Processing: The Rayleigh distribution plays a crucial role in modelling noise in applications like MRI image processing. It aids in characterizing noise characteristics and developing effective denoising techniques.

b) Sea State Analysis: In oceanography and wave modelling, the Rayleigh distribution is employed to describe the distribution of wave heights in random sea states. This assists in predicting extreme wave events.

c) Wireless Communication: The Rayleigh distribution is a common choice for modelling the magnitude of received signals in fading wireless communication channels due to multipath propagation.

d) Wind Speed Analysis: In meteorology and wind engineering, the Rayleigh distribution is applied to model the distribution of wind speeds. This provides valuable insights into statistical properties and aids in estimating extreme wind events, among other applications.

This real-world application of statistical techniques offers valuable insights into the behavior of these distributions and holds potential utility for modelling or prediction purpose. Overall, the study provides a comprehensive analysis of multiple distributions and their parameters, potentially impacting a wide range of applications across various fields.

4. CONCLUSION

In this study, we have addressed the problem of estimating parameters for two similar continuous probability distributions, such as Lomax, Weibull, Rayleigh, and Gamma distributions, utilizing the method of Maximum Likelihood Estimation (MLE). Due to the absence of closed-form expressions for the ML estimates, we employed a numerical technique known as the Newton-Raphson method to approximate the ML estimates of the common parameter by solving the system of equations for different distributions. Following the calculation of the ML estimates for each distribution, we proposed 95% asymptotic confidence intervals for each parameter of every distribution using the Fisher information matrix. Additionally, the study conducted simulations for each distribution with varying sample sizes, estimating the values of the parameters, bias, and mean squared error (MSE). This comprehensive analysis allowed for a thorough examination of the behavior of each distribution under different conditions.

Based on the simulation data, we can conclude that increasing the sample size reduces both the bias and mean square error for each estimator. It has been observed that for fixed sample sizes and fixed shape parameters, the common scale parameter of these distributions increases, resulting in decreased bias and mean square error values.

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