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COMPLEMENTARY OF CLASSICAL MEANS WITH RESPECT TO HERON MEAN AND THEIR SCHUR CONVEXITIES

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A B S T R A C T

In this paper, the complementary of arithmetic mean, geometric mean, harmonic mean and contra harmonic mean with respect to Heron mean are defined. Further, by finding the partial derivatives developed the Schur convexity and Schur geometric convexity results.

1. INTRODUCTION

The well-known means in literature namely; Arithmetic mean, Geometric mean and Harmonic mean are presented by Pappus of Alexandria [Bullen, 2003]. In Pythagorean school on the basis of proportion, 10 Greek means are defined, among these means the well-known in literature are given as follows: For two positive real numbers a and b; $A(a,b) = \frac{a+b}{2}$; $G(a,b) = \sqrt{ab}$; $H(a,b) = \frac{2ab}{a+b}$ and $C(a,b) = \frac{a^2+b^2}{a+b}$. These are respectively called Arithmetic mean (A.M), Geometric mean (G.M), Harmonic mean (H.M) and Contraharmonic mean (C.M). In 1958, C. Gini introduced Complementary means and G.Toader in 1991 proposed a generalization of complementariness and inversion (Toader, S. & Toader, G. 2005). In 1923, Issai Schur introduced the concept of Schur convexity, which has applications in linear regression, analytic inequalities, Gamma functions, graphs and matrices, stochastic orderings, combinatorial optimization, information theoretic topics, reliability and related fields.

2. LITERATURE REVIEW

The Schur geometrical convexity and concavity of the extended and other mean values were discussed by (Chu, Y. M. et al, 2008, Shi, H. N. et al, 2010, Murali, K. & Nagaraja, K. M. 2013, 2016, Janardhana, L. et al, 2017)

Results on Schur convex and concave functions studied by (Elezovic, N. et al, 1998, Janardhana, L. et al, 2017a,

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2017b, Lokesh, et al, 2010a, 2010b, 2011, Marshall, A. M. & Olkin, I. 1979, Nagaraja, K. M. et al, 2008, 2010, 2011, 2013, 2014a, 2014b, 2014c, 2017, Naveen Kumar, B. et al, 2014, Sampath Kumar, R. et al., 2017a, 2017b).

The Schurharmonic convexity and concavityresults were discussed in (Nagaraja, K. M. et al., 2013a, 2013b). This paper was developed based on the above literature survey.

3. DEFINITIONS AND LEMMAS

Definition 1: (Toader, S. & Toader, G. 2005) A mean N is called complementary to M with respect to P(or P-complementary to M) if it verifies $P(M,N)=P$, it is denoted by $M^{(P)}=N$.

1. The complementary of geometric mean with respect to Heron mean is denoted by $G^{(He)}$ and is given by

$$G^{(He)}=\frac{1}{2}\left[2a+2b+\sqrt{ab}-\sqrt{4(a+b)\sqrt{ab}+ab}\right].$$

2. The complementary of arithmetic mean with respect to Heron mean is denoted by $A^{(He)}$ and is given by

$$A^{(He)}\frac{1}{4}\left[3a+3b+4\sqrt{ab}-\sqrt{(a+b)(5a+5b+8\sqrt{ab})}\right].$$

3. The complementary of harmonic mean with respect to Heron mean is denoted by $H^{(He)}$ and is given by

$$H^{(He)}=a+b+\sqrt{ab}-\frac{ab}{a+b}-\frac{\sqrt{2ab(a+b)^2+2\sqrt{ab}(a+b)-3ab}}{a+b}$$

4. The complementary of contra harmonic mean with respect to Heron mean is denoted by $C^{(He)}$ and is given by

$$C^{(He)}=2\left(a+b+\sqrt{ab}\right)-\frac{a^2+b^2}{a+b}-\frac{\sqrt{[(a^2+b^2)+8ab+4\sqrt{ab}(a+b)](a^2+b^2)}}{a+b}$$

Definition 2: [A. M. Marshall and I. Olkin, 1979]
Let $x=(x_1,x_2,\dots,x_n)\ and\ y=(y_1,y_2,\dots,y_n)\epsilon R^n$
(i) x is majorized by y, (in symbol x ≼ y). If$\sum_{i=1}^{k}x_i\le\sum_{i=1}^{k}y_i$, $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i$where $x_1\ge x_2,\ge\dots,\ge x_n$and $y_1\ge y_2\ge\dots\ge y_n$ are rearrangements of x and y in descending order.(ii) $\Omega\subseteq R^n$is called a convex set if $(\alpha x_1+\beta y_1\dots\dots\alpha x_n+\beta y_n)$for every x, y $\epsilon\Omega$, where $\alpha,\beta\epsilon[0,1]$with $\alpha+\beta=1$.

Let $\Omega\subseteq R^n$ the function $\varphi:\Omega\longrightarrow R$ is said to be a Schur convex function on Ω if $x\le\ y$ on Ω implies $\varphi(x)\le\varphi(y)$. $\varphi$is said to be a Schur concave function on Ω if and only if - φ is Schur convex.

Definition 3: Let $x=(x_1,x_2,\dots,x_n)\ and\ y=(y_1,y_2,\dots,y_n)\epsilon {R^n}_+$ $\Omega\subseteq R^n$ is called geometrically convex set if for everyx, y $\epsilon\varphi$, where $\alpha,\beta\epsilon[0,1]$with $\alpha+\beta=1$.
Let $\Omega\subseteq R^n$, the function $\varphi:\Omega\longrightarrow R_+$ is said to be Schur geometrically convex function on Ω if $(lnx_1,lnx_2,\dots,lnx_n)\prec(lny_1,lny_2,\dots,lny_n)$ on Ω implies
$\varphi(x)\le\varphi(y)$. Then φ is said to be a Schur geometrically concave function on φ if and only if $-\varphi$ is Schur geometrically convex.

Definition 4: $\Omega\subseteq R^n$is called symmetric set if x $\epsilon\Omega$ implies $Px\epsilon\Omega$ for every nxn permutation matrix P.

The function $\varphi:\Omega\longrightarrow R$ is called symmetric if for every permutation matrix P, φ(Px)= $\varphi(x)$ for all $x\epsilon\Omega$.

Definition 5: Let $\Omega\subseteq R^n$ $\varphi:\Omega\longrightarrow R$ be symmetric and convex function, then φ is Schur convex on Ω.

Lemma 1: Let $\Omega\subseteq R^n$ be symmetric with non-empty interior geometrically convex set and let $\varphi:\Omega\longrightarrow R_+$ be continuous on Ω and differentiable in Ω^0. If
φ is symmetric on Ω and

$$(x_1-x_2)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}\right)\ge 0(\le 0)$$

$$(lnx_1-lnx_2)\left(x_1\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{x_2\partial x_2}\right)\ge 0(\le 0)$$

$$(x_1-x_2)\left({x_1}^2\frac{\partial\varphi}{\partial x_1}-{x_2}^2\frac{\partial\varphi}{\partial x_2}\right)\ge 0(\le 0)$$

is a Schur convex (concave), Schur-geometrically convex(concave) and Schur-harmonically convex (concave) function respectively.

4. MAIN RESULTS

Theorem 1: For $a,b>0$, the complementary of arithmetic mean with respect to Heron mean is Schur concave.

Proof: The complementary of arithmetic mean with respect to Heron mean is given by;

$$A^{He}=\frac{1}{4}\left[3a+3b+4\sqrt{ab}-\sqrt{(a+b)\{5a+5b+8\sqrt{ab}\}}\right]$$

The partial derivatives of A^{He} with respect to a and b are given by

$$\frac{\partial A^{He}}{\partial a}=\frac{1}{4}\left[3+\frac{2b}{\sqrt{ab}}-\frac{\frac{4b(a+b)}{\sqrt{ab}}+10(a+b)+8\sqrt{ab}}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

and

$$\frac{\partial A^{He}}{\partial b}=\frac{1}{4}\left[3+\frac{2a}{\sqrt{ab}}-\frac{\frac{4a(a+b)}{\sqrt{ab}}+10(a+b)+8\sqrt{ab}}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Consider,

$$\frac{\partial A^{He}}{\partial a}-\frac{\partial A^{He}}{\partial b}=\frac{1}{2}\left[\frac{b-a}{\sqrt{ab}}-\frac{\frac{(b-a)(a+b)}{\sqrt{ab}}}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Then,

$$(a-b)\left[\frac{\partial A^{He}}{\partial a}-\frac{\partial A^{He}}{\partial b}\right]=\frac{(a-b)^2}{2\sqrt{ab}}\left[\frac{a+b-\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}{\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Clearly, $0<4(a+b)^2+8\sqrt{ab}(a+b)$, adding $(a+b)^2$ on both sides, then

$$a+b<\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}$$

Thus, $(a-b)\left[\frac{\partial A^{He}}{\partial a}-\frac{\partial A^{He}}{\partial b}\right]<0.$
Hence the proof of the theorem 1.

Theorem 2: For $a,b>0$, the complementary of arithmetic mean with respect to Heron mean is Schur geometrically convex.

Proof: The complementary of arithmetic mean with respect to Heron mean is given by;

$$A^{He}=\frac{1}{4}\left[3a+3b+4\sqrt{ab}-\sqrt{(a+b)\{5a+5b+8\sqrt{ab}\}}\right]$$

The partial derivatives of A^{He} with respect to a and b are given by

$$\frac{\partial A^{He}}{\partial a}=\frac{1}{4}\left[3+\frac{2b}{\sqrt{ab}}-\frac{\frac{4b(a+b)}{\sqrt{ab}}+10(a+b)+8\sqrt{ab}}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

and

$$\frac{\partial A^{He}}{\partial b}=\frac{1}{4}\left[3+\frac{2a}{\sqrt{ab}}-\frac{\frac{4a(a+b)}{\sqrt{ab}}+10(a+b)+8\sqrt{ab}}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Consider,

$$a\frac{\partial A^{He}}{\partial a}-b\frac{\partial A^{He}}{\partial b}=\frac{1}{4}\left[3(a-b)-\frac{5(b-a)(a+b)+4\sqrt{ab}(b-a)}{2\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Then,

$$(lna-lnb)\left[a\frac{\partial A^{He}}{\partial a}-b\frac{\partial A^{He}}{\partial b}\right]=\frac{(lna-lnb)(a-b)}{4}\left[3-\frac{5(a+b)+4\sqrt{ab}}{\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}}\right]$$

Clearly, $3\sqrt{5(a+b)^2+8\sqrt{ab}(a+b)}-(5(a+b)+4\sqrt{ab})>0$

Thus, $(lna-lnb)\left[a\frac{\partial A^{He}}{\partial a}-b\frac{\partial A^{He}}{\partial b}\right]>0.$
Hence the proof of the theorem 2.

Theorem 3: For $a,b>0$, the complementary of geometric mean with respect to Heron mean is Schur convex.

Proof: The complementary of geometric mean with respect to Heron mean is given by

$$G^{He}=\frac{1}{2}\left[2a+2b+\sqrt{ab}-\sqrt{4(a+b)\sqrt{ab}+ab}\right]$$

The partial derivative of G^{He} with respect to a and b are given by

$$\frac{\partial G^{He}}{\partial a}=\frac{1}{4}\left[4+\frac{b}{\sqrt{ab}}-\frac{b\left(\sqrt{ab}+6a+2b\right)}{\sqrt{ab}\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

and

$$\frac{\partial G^{He}}{\partial b}=\frac{1}{4}\left[4+\frac{a}{\sqrt{ab}}-\frac{a\left(\sqrt{ab}+6b+2a\right)}{\sqrt{ab}\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

Then,

$$(a-b)\left[\frac{\partial G^{He}}{\partial a}-\frac{\partial G^{He}}{\partial b}\right]=\frac{(a-b)^2}{4\sqrt{ab}}\left[\frac{\sqrt{ab}+2(a+b)-\sqrt{ab+4\sqrt{ab}(a+b)}}{\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

Clearly, $\sqrt{ab}+2(a+b)-\sqrt{ab+4\sqrt{ab}(a+b)}>0$

Thus, $(a-b)\left[\frac{\partial G^{He}}{\partial a}-\frac{\partial G^{He}}{\partial b}\right]>0.$

Hence the proof of the theorem 3.

Theorem 4: For $a,b>0$, the complementary of geometric mean with respect to Heron mean is Schur geometrically convex.

Proof: The complementary of geometric mean with respect to Heron mean is given by

$$G^{He}=\frac{1}{2}\left[2a+2b+\sqrt{ab}-\sqrt{4(a+b)\sqrt{ab}+ab}\right]$$

The partial derivative of G^{He} with respect to a and b are given by

$$\frac{\partial G^{He}}{\partial a}=\frac{1}{4}\left[4+\frac{b}{\sqrt{ab}}-\frac{b\left(\sqrt{ab}+6a+2b\right)}{\sqrt{ab}\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

and

$$\frac{\partial G^{He}}{\partial b}=\frac{1}{4}\left[4+\frac{a}{\sqrt{ab}}-\frac{a\left(\sqrt{ab}+6b+2a\right)}{\sqrt{ab}\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

Then, $(lna-lnb)\left[a\frac{\partial G^{He}}{\partial a}-b\frac{\partial G^{He}}{\partial b}\right]=$

$$\frac{(lna-lnb)(a-b)}{4}\left[4-\frac{4\sqrt{ab}}{\sqrt{ab+4\sqrt{ab}(a+b)}}\right]$$

Clearly, $4\sqrt{ab+4\sqrt{ab}(a+b)}-4\sqrt{ab}>0$

Thus, $(lna-lnb)\left[a\frac{\partial G^{He}}{\partial a}-b\frac{\partial G^{He}}{\partial b}\right]>0.$

Hence the proof of the theorem 4.

Theorem 5: For $a,b>0$, the complementary of contra harmonic mean with respect to Heron mean is Schur concave.

Proof: The complementary of contra harmonic mean with respect to Heron mean is given by

$$C^{He}=2a+2b+2\sqrt{ab}-\frac{a^2+b^2}{a+b}-\frac{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}{a+b}$$

The partial derivatives of C^{He} with respect to a and b are given by

$$\frac{\partial C^{He}}{\partial a}=2+\frac{b}{\sqrt{ab}}-\frac{2a}{a+b}+\frac{(a^2+b^2)}{(a+b)^2}+\frac{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}{(a+b)^2}-\frac{(a^2+b^2)\left[4a+8b+\frac{2b(a+b)}{\sqrt{ab}}+4\sqrt{ab}\right]}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}-\frac{\left[4a+8b+\frac{2b(a+b)}{\sqrt{ab}}+4\sqrt{ab}\right]+16a^2b+8a(a+b)\sqrt{ab}}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}$$

and

$$\frac{\partial C^{He}}{\partial b}=2+\frac{a}{\sqrt{ab}}-\frac{2b}{a+b}+\frac{(a^2+b^2)+\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}{(a+b)^2}-\frac{(a^2+b^2)\left[4b+8a+\frac{2a(a+b)}{\sqrt{ab}}+4\sqrt{ab}\right]}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}-\frac{16ab^2+8b(a+b)\sqrt{ab}}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}$$

Consider,

$$\frac{\partial C^{He}}{\partial a}-\frac{\partial C^{He}}{\partial b}=\frac{b-a}{\sqrt{ab}}+\frac{2(b-a)}{a+b}+\frac{(a^2+b^2)\left[4(a-b)+\frac{2(a-b)(a+b)}{\sqrt{ab}}\right]}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}+\frac{(b-a)\left(16ab+8\sqrt{ab}(a+b)\right)}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}$$

Then, $(a-b)\left[\frac{\partial C^{He}}{\partial a}-\frac{\partial C^{He}}{\partial b}\right]=-(a-b)^2\left[\frac{1}{\sqrt{ab}}+\frac{2}{a+b}+\frac{-4(a^2+b^2)-\frac{2(a^2+b^2)(a+b)}{\sqrt{ab}}+16ab+8\sqrt{ab}(a+b)}{2(a+b)\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}\right]$

$$=\frac{-(a-b)^2\left(\sqrt{a}+\sqrt{b}\right)^2}{(a+b)\sqrt{ab}}\left[1+\frac{4ab-a^2-b^2}{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}\right]$$

Clearly,

$\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}+4ab-a^2-b^2>0$

Thus, $(a-b)\left[\frac{\partial C^{He}}{\partial a}-\frac{\partial C^{He}}{\partial b}\right]<0.$

Hence the proof of the theorem 5.

Theorem 6: For $a,b>0$, the complementary of contra harmonic mean with respect to Heron mean is Schur geometrically concave.

Proof: The complementary of contra harmonic mean with respect to Heron mean is given by

$$C^{He}=2a+2b+2\sqrt{ab}-\frac{a^2+b^2}{a+b}-\frac{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}{a+b}$$

The partial derivatives of C^{He} with respect to a and b are given by (**refer theorem 5**)

Consider,

$$a\frac{\partial C^{He}}{\partial a}-b\frac{\partial C^{He}}{\partial b}=\frac{(b-a)[-a^2-b^2]}{(a+b)^2}+\left[\frac{b-a}{(a+b)^2}\right]$$
$$\left[\frac{(a^2+b^2)(a^2+b^2+4ab)}{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}+\frac{2\sqrt{ab}(a+b)(3a^2+3b^2+8ab)}{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}\right]$$

Then,

$$(lna-lnb)\left[a\frac{\partial C^{He}}{\partial a}-b\frac{\partial C^{He}}{\partial b}\right]=\frac{(lna-lnb)(b-a)}{(a+b)^2}$$
$$\left[\frac{(a^2+b^2)(a^2+b^2+4ab)}{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}\right.$$
$$\left.\frac{2\sqrt{ab}(a+b)(3a^2+3b^2+8ab)}{\sqrt{(a^2+b^2)^2+(a^2+b^2)\left[8ab+4\sqrt{ab}(a+b)\right]}}-a^2-b^2\right]$$

Clearly, above expression is positive.

Thus, $(lna-lnb)\left[a\frac{\partial C^{He}}{\partial a}-b\frac{\partial C^{He}}{\partial b}\right]< 0.$

Hence the proof of the theorem 6.

Theorem 7: For $a,b>0$, the complementary of harmonic mean with respect to Heron mean is Schur concave.

Proof: The complementary of harmonic mean with respect to Heron mean is given by

$$H^{He}=a+b+\sqrt{ab}-\frac{ab}{a+b}-\frac{\sqrt{ab\left[2(a+b)^2+2(a+b)\sqrt{ab}-3ab\right]}}{a+b}$$

The partial derivatives of H^{He} with respect to a and b are given by

$$\frac{\partial H^{He}}{\partial a}=1+\frac{b}{2\sqrt{ab}}-\frac{b}{a+b}+\frac{ab+\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}{(a+b)^2}$$
$$-\frac{ab\left(\frac{b(a+b)}{\sqrt{ab}}+4(a+b)+2\sqrt{ab}-3b\right)}{2(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}-\frac{b\left(2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right)}{2(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}$$

and

$$\frac{\partial H^{He}}{\partial b}=1+\frac{a}{2\sqrt{ab}}-\frac{a}{a+b}+\frac{ab+\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}{(a+b)^2}$$
$$-\frac{ab\left(\frac{a(a+b)}{\sqrt{ab}}+4(a+b)+2\sqrt{ab}-3a\right)}{2(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}-\frac{a\left(2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right)}{2(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}$$

Then,

$$(a-b)\left(\frac{\partial H^{He}}{\partial a}-\frac{\partial H^{He}}{\partial b}\right)=\left[\frac{(a-b)^2}{2\sqrt{ab}(a+b)}\right]\times$$
$$\left[2\sqrt{ab}-a-b+\frac{2(a+b)^2+3\sqrt{ab}(a+b)-6ab}{\sqrt{2(a+b)^2+2\sqrt{ab}(a+b)-3ab}}\right]$$

Clearly,

$$2\sqrt{ab}-a-b+\frac{2(a+b)^2+3\sqrt{ab}(a+b)-6ab}{\sqrt{2(a+b)^2+2\sqrt{ab}(a+b)-3ab}}>0$$

Thus $(a-b)\left(\frac{\partial H^{He}}{\partial a}-\frac{\partial H^{He}}{\partial b}\right)<0.$

Hence the proof of the theorem 7.

Theorem 8: For $a,b>0$, the complementary of harmonic mean with respect to Heron mean is Schur geometrically convex.

Proof: The complementary of harmonic mean with respect to Heron mean is given by

$$H^{He}=a+b+\sqrt{ab}-\frac{ab}{a+b}-\frac{\sqrt{ab\left[2(a+b)^2+2(a+b)\sqrt{ab}-3ab\right]}}{a+b}$$

The partial derivatives of H^{He} with respect to a and b are given by (**refer theorem7**)

Consider,

$$(lna-lnb)\left(a\frac{\partial H^{He}}{\partial a}-b\frac{\partial H^{He}}{\partial b}\right)=(lna-lnb)(a-b)\times$$
$$\left[\left(1+\frac{ab+\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}{(a+b)^2}\right)-\left(\frac{2ab(a+b)+ab\sqrt{ab}}{(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}\right)\right]$$

Clearly,

$$\left(1+\frac{ab+\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}{(a+b)^2}\right) - \left(\frac{2ab(a+b)+ab\sqrt{ab}}{(a+b)\sqrt{ab\left[2(a+b)^2+2\sqrt{ab}(a+b)-3ab\right]}}\right) > 0$$

Thus, $(lna - lnb)\left(a\frac{\partial H^{He}}{\partial a} - b\frac{\partial H^{He}}{\partial b}\right) > 0.$
Hence the proof of the theorem 8.

4. CONCLUSION

The results discussed in this paper have applications in linear regression, analytic inequalities, Gamma functions, graphs and matrices, stochastic orderings, combinatorial optimization, information theoretic topics, reliability and related fields.

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